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ČECH AND STEENROD HOMOTOPY AND THE QUIGLEY EXACT COUPLE IN STRONG SHAPE AND PROPER HOMOTOPY THEORY

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In 1970, Quigley introduced two definitions of 'homotopy groups' for metric compacta and showed that there was an exact couple relating them and the shape groups of Borsuk, [15]. Recently his theory has been studied by various authors: Kodama and Koyama [11] proved a Hurewicz isomorphism theory between Quigley's 'approaching group' and Steenrod homology; Watanabe [16] proved a Milnor exact sequence for the same groups (in fact this is part of Quigley's exact couple [15]) Koyama, Ono and Tsuda [12] have used Quigley's theory to characterise S^{n+1}-movable spaces and Cathey [6], in his thesis has shown that Quigley's approaching category is, in fact, a strong shape theory à la Edwards–Hastings [8].

Using this last result one can ask for a definition of both of the Quigley groups within the language of the prohomotopy theory of Edwards and Hastings. Once one has this, one notices that Quigley's definition of the approaching groups translates as the Steenrod homotopy groups of Edwards and Hastings, and his definition of 'inward group' translates as the Brown–Grossman homotopy groups. These latter were introduced by Brown [2] as being a useful tool in proper homotopy theory. This they have indeed proved to be (see Brown [3], Brown–Tucker [5], Brown–Messer [4], and Chipman [7] for some calculations of the proper fundamental group). The translation into prohomotopy theory was made by Grossman [9].

In this note we shall assume known the basic methods of passing from proper homotopy and shape theory into prohomotopy theory. These methods are amply treated in the lecture notes of Edwards and Hastings [8] but for completeness we will give a brief sketch.

If X is a locally compact, σ-compact, Hausdorff space, one can construct an inverse system of spaces, the end of X, by putting

$$\varepsilon(X) = \{\text{closure}(X - A) \mid A \text{ a compactum in } X\}$$

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with the obvious inclusion maps as inclusions. A continuous function $f:X\to Y$ of locally compact spaces is easily seen to be proper if and only if f induces a map $\varepsilon(f):\varepsilon(X)\to\varepsilon(Y)$ which makes the diagram

$$\begin{tikzcd} \varepsilon(X) \arrow[r, "\varepsilon(f)"] & \varepsilon(Y) \\ X \arrow[u] \arrow[r, "f"] & Y \arrow[u] \end{tikzcd}$$

commute. The functor ε gives a full embedding of the proper category (at infinity), $P_{\sigma,\infty}$, into pro-Top.

If now X is a (pointed) compactum embedded in the pseudo-interior of I^{ω}, we can pick an inclusion system, U_X, of (pointed) ANR-neighbourhoods of X in I^{ω}. This assignment will not be functorial. (There are various ways of getting around this, for instance the Čech and Vietoris constructions used in [13].) However if one passes to H_0(pro-Top) by inverting 'level weak equivalences' (cf. [8, Ch. 5]), then one does obtain a functor from the category of compacta to H_0(pro-Top). The category obtained from the category of compacta by defining

$$s\text{-Sh}(X,Y)=H_0(\text{pro-Top})\{U_X,U_Y\}$$

is called the Edwards–Hastings strong shape category.

Returning to the proper category, we can again pass to H_0(pro-Top). One finds that ε induces an embedding of the homotopy category $H_0(P_{\sigma,\infty})$ into H_0(pro-Top).

We shall sketch the equivalence of Quigley's definitions with the Steenrod and Brown–Grossman definitions and then shall give a proof of a generalised form of the Quigley exact couple. Following that we shall analyse this exact couple obtaining as a consequence, amongst other things a quick proof of Watanabe's form of the Milnor exact sequence for the approaching groups. (In fact the Watanabe exact sequence is part of the Bousfield–Kan Spectral sequence, [1]; that this is so follows immediately from the identification of the approaching groups as Steenrod groups.)

1. The approaching groups and Steenrod groups

Quigley [15] makes the following definitions:

Let (X,x) be a pointed metric compactum embedded in the Hilbert cube, I^{ω}. We denote $\mathbb{R}^+$ the nonnegative real half line.

A continuous mapping ξ from $\mathbb{R}^+\times S^n$ to I^{ω} is said to be an approaching n-mapping of (X,x) if and only if

(i) $\xi(\mathbb{R}^+\times\{p_0\})=\{x\}$ where $p_0=(1,0,\dots,0)\in S^n$,

(ii) given a neighbourhood V of X in I^{ω} there is an $r\in\mathbb{R}^+$ such that $\xi([r,\infty)\times S^n)\subset V$.

If ξ, ξ' are approaching n-mappings of (X,x) then we say that ξ is approaching homotopic to ξ' if and only if there is a continuous map $\Phi: \mathbb{R}^+ \times S^n \times I \to I^\omega$ such that

(i) $\Phi \mid \mathbb{R}^+ \times S^n \times \{0\} = \xi$, $\Phi \mid \mathbb{R}^+ \times S^n \times \{1\} = \xi'$,

(ii) $\Phi(\mathbb{R}^+ \times \{p_0\} \times I) = \{x\}$,

(iii) given neighbourhood V of X there is an $r \in \mathbb{R}^+$ such that $\Phi([r, \infty) \times S^n \times I) \subset V$.

In this case we write $\Phi: \xi \simeq \xi'$ (approaching). Approaching homotopy is an equivalence relation on the set of all approaching n-mappings of (X,x) and the set of classes of approaching n-mappings of (X,x) gives a group $\underline{\underline{\pi}}_n(X,x)$, $n>0$ (and a set $\underline{\underline{\pi}}_0(X,x)$ for $n=0$).

Remark. If one tries to use this definition in proper homotopy theory (e.g. by looking at $I^\omega - X$) one has the difficulty that $\xi(\mathbb{R}^+ \times \{p_0\}) = \{x\}$ which is not, of course, in $I^\omega - X$. In proper homotopy theory the usual way around this type of difficulty is to replace a base point by a 'base ray' $*: \mathbb{R}^+ \to M$ (cf. Brown [2]) thus the analogue of the approaching groups in the proper homotopy theory of a simplicial complex M with base ray, $*$, would seem to be the set of proper homotopy classes of proper maps from the cylinder $S^n \times \mathbb{R}^+$ to M which 'send' $\{p_0\} \times \mathbb{R}^+$ to $*$ (again compare the definition of Brown's proper homotopy groups [2]). The usual transition (via ends, see Edwards–Hastings [8, §6]) from the proper homotopy category to the pro-homotopy category (again see Edwards–Hastings [8]) gives a definition which is the same as that given in [8] for the strong homotopy groups of a pointed prosimplicial set. As these are the prohomotopical analogue of Quigley's groups, (see below), it would seem that this definition of proper homotopy groups at an end, $*$ gives the correct analogue of Quigley's approaching groups in this context.

Cathey in his thesis [6], shows that the approaching category of Quigley is equivalent to the Edwards–Hastings strong shape category [8]. The Quigley approaching groups are given by

$$\underline{\underline{\pi}}_n(X,x) \cong [(S_n, p_0), (X,x)]$$

where [,] here denotes the 'hom-set' in the approaching category. Using the results of Edwards–Hastings [8, p. 231] and Cathey [6] we must therefore have

$$\underline{\underline{\pi}}_n(X,x) \cong [(S^n, p_0), (\underline{V},x)]$$

where $(\underline{V},x)$ in pro-Top is an inclusion system of (pointed) ANR-neighbourhoods of (X,x) in I^ω.

Thus the nth approaching group of Quigley is isomorphic to the nth strong homotopy group of an associated pointed prospace (e.g. the geometric realization of the pointed Vietoris complex [13] of (X,x)).

One thus has a wider definition of Quigley's approaching groups applicable to any pointed topological space. In general one has a Bousfield–Kan spectral sequence:

$$\varprojlim^{(p)} \pi_q((\underline{X},x)) \Rightarrow \pi_n(\mathrm{holim}(\underline{X},x))$$

for a pointed prosimplicial set $(\underline{X},x)$ and a natural isomorphism

$$\pi_n(\mathrm{holim}(\underline{X},x)) \cong [(S^n,p_0),(\underline{X},x)] \cong \underline{\underline{\pi}}_n(\underline{X},x).$$

If (X,x) is a metric compactum, $\varprojlim^{(p)} \pi_q(V_k,x)=0$ for all $p>1$ and the spectral sequence collapses to give

$$0 \to \varprojlim^{(1)} \pi_{q+1}(V_k,x) \to \underline{\underline{\pi}}_q(X,x) \to \varprojlim \pi_q(V_k,x) \to 0$$

which is the homotopy Milnor sequence for the approaching groups given by Watanabe [16]. We shall give a different proof of this result later.

If we suspend both terms in $[(S^n,p_0),(\underline{X},x)]$ and take direct limits one obtains the stable strong homotopy groups of $(\underline{X},x)$ and, using a spectrum E, we can also get E-homology groups à la Edwards–Hastings [8, p. 252]. We shall assume this to be done. In the case of a pointed metric compactum (X,x) these are generalised Steenard homology groups (again see [8]).

Remark. We propose to give all these groups – homotopy, stable homotopy, and homology groups – the label 'Steenrod', likewise their prohomotopical and proper homotopy analogues. This extension of the term is then invariant under the transitions between these contexts; this will simplify considerably the exposition.

We have thus identified the approaching homotopy groups as being 'really' the Steenrod homotopy groups. This puts the results of Kodama and Koyama [11] into perspective. If one looks for a Hurewicz isomorphism theorem for Steenrod homology one needs 'Steenrod homotopy groups', however these are essentially the same as Quigley's approaching groups. The conditions imposed by Kodama and Koyama appear to be natural from this view point: they are precisely those conditions which are known to guarantee that the relevant Milnor sequences in homotopy and homology are isomorphic via the Hurewicz maps.

2. The inward groups and Brown–Grossman groups

We next turn to the definition of the second sequence of groups introduced, in [15], by Quigley. We shall perform the same transition to a prohomotopical situation and shall compare with the prohomotopy groups defined by Grossman [9], which follow the transition from proper homotopy theory to prohomotopy theory, starting from the definition of Brown [2].

A continuous mapping ξ from $\mathbb{N} \times S^n$ to I^ω is said to be an inward n-mapping of a pointed metric compactum (X,x) in I^ω if and only if

(i) $\xi(\mathbb{N}^+ \times \{p_0\}) = \{x\}$,

(ii) given a neighbourhood V of X there is a $j_0 \in \mathbb{N}$ that $\xi(\{j\} \times S^n) \subset V$ for all $j \geq j_0$.

The definition of an inward homotopy of inward n-mappings is, we hope, self evident.

First we note that an inward n-mapping induces a map from the infinite wedge $\bigvee S^n$ to I^ω and likewise such a map ξ gives an inward n-mapping provided it satisfies the condition:

Given a neighbourhood V of X there is a $j_0 \in N$ such that $\xi(\bigvee_{k \geq j_0} S^n) \subset V$.

Using as before an amalgam of ideas of Edwards–Hastings and Cathey, we define a prospace $\underline{S}^n$ indexed by the positive integers

$$\underline{S}^n(k) = \bigvee_{j \geq k} S_j^n$$

where each S_j^n is a copy of S^n, the structure maps $\underline{S}^n(k+1) \to S^n(k)$ are the obvious inclusions and note that approaching maps from $\bigvee(S^n, p_0)$ to (X,x) correspond in a one–one fashion to the elements of the set $[(\underline{S}^n, p_0), (\underline{V}, x)]$ (with $(\underline{V}, x)$ as before). Thus we obtain a prohomotopical description of the inward groups.

These groups have been studied in some depth by Grossman [9] and Chipman [7]. In fact the proper homotopy groups of Brown [2] are the original proper homotopy analogue of these groups and they have been very successfully applied in the proper homotopy theory of 3-manifolds (see the references in the introduction). We propose to give them the name of Brown–Grossman homotopy groups. (Clearly there are stable forms also 'E-homology groups' defined in the obvious way.) We shall denote them by $\underline{\pi}_n$ etc. following Brown [2]. Grossman [9] proves that for a tower of fibrations $\{X_s\}$, $\underline{\pi}_q(\{X_s\})$ can be calculated to be $\varprojlim I\{\pi_q(X_s)\}$ where I denotes the reduced product functor. Thus if G is a group, $I(G)$ is the quotient of the product $G^{\mathbb{N}}$ by the relation

$$\underline{g} \equiv \underline{g}' \quad \text{if and only if} \quad \{i \mid g(i) \neq g'(i)\} \text{ is finite.}$$

(See also Porter [14] for more details of the connections between reduced products and inverse limits.)

3. The Quigley exact couple

In [15], Quigley showed that, for a pointed metric compactum (X,x), there is, in our notation, an exact sequence

$$0 \to \check{\pi}_n(X,x) \to \underline{\pi}_n(X,x) \to \underline{\pi}_n(X,x) \to \underline{\underline{\pi}}_{n-1}(X,x) \to \check{\pi}_{n-1}(X,x) \to 0$$

for each $n \geq 1$, where each of the maps has a precise geometric interpretation.

Putting these sequences together end-to-end gives an exact couple

$$\begin{tikzcd}
\underline{\pi}_*(X,x) \arrow[rr, "\beta"] & & \underline{\pi}_*(X,x) \arrow[dl, "\gamma"] \\
& \underline{\underline{\pi}}_*(X,x) \arrow[ul, "\alpha"] &
\end{tikzcd}$$

where γ has degree -1. It is this that we call the Quigley exact couple.

In a second article (apparently unpublished) Quigley gives a similar homology exact couple and analyses the maps in it to a greater extent than in [15].

We shall prove the existence of this exact couple for homotopy in a more general setting, namely that of prohomotopy theory. We shall also pass to the stable prohomotopy category and there our result will give stable homotopy and E-homology forms of this Quigley exact couple. We thus obtain a generalisation of Quigley's unpublished result.

First let us recall that given two prospaces, $\underline{X}$ and $\underline{Y}$, indexed by the same indexing category, a level cofibration from $\underline{X}$ to $\underline{Y}$ is a level map which is a cofibration at each level (cf. Edwards–Hastings [8]). Given such a cofibration one gets a long cofibration sequence. In general for any $\underline{f}:\underline{X}\to\underline{Y}$ by reindexing and use of mapping cylinders one can replace $\underline{f}$ by a level cofibration, thus for any $\underline{f}$ and $\underline{Z}$ one has an exact sequence,

$$\to[\Sigma^r\underline{Y},\underline{Z}]\to[\Sigma^r\underline{X},\underline{Z}]\to[\Sigma^{r-1}\underline{C}_f,\underline{Z}]\to\cdots$$
$$\cdots\to[\Sigma\underline{X},\underline{Z}]\to[\underline{C}_f,\underline{Z}]\to[\underline{Y},\underline{Z}]\to[\underline{X},\underline{Z}]$$

where $\underline{C}_f$ is the mapping cone of $\underline{f}$, (cf. Edwards–Hastings [8, p. 101]).

We shall apply this construction in the pointed case to the folding map in pro(Top.),

$$\underline{S}^0 \xrightarrow{f} S^0$$

given by, in level k, the usual folding map.

$$f_k: \bigvee_{j\ge k} S_j^0 \to S^0.$$

Note first that $\Sigma^r\bigvee_{j\ge k} S_j^0 \xrightarrow{\Sigma^r f} \Sigma^r S^0$ is, in fact, the folding map from $\bigvee_{j\ge k} S_j^r$ to S^r.

For any pointed prospace $(\underline{X},x)$, we have a long exact sequence

$$\to[\underline{S}^r,\underline{X}]\to[\Sigma^{r-1}\underline{C}_f,\underline{X}]\to[S^{r-1},\underline{X}]\to[\underline{S}^{r-1},\underline{X}]$$
$$\to[\underline{C}_f,\underline{X}]\to[S^0,\underline{X}]\to[\underline{S}^0,\underline{X}].$$

We can identify $[S^r,\underline{X}]$ as $\underline{\underline{\pi}}_r(X)$, $[\underline{S}^r,\underline{X}]$ as $\underline{\pi}_r(X)$ and we are left to investigate $\Sigma^{r-1}\underline{C}_f$ and the morphisms.

To calculate $\underline{C}_f$ we need to turn f into a cofibration; this can be done 'levelwise'. Thus it suffices to look at the folding map f_k. The calculation here is, of course, easily made.

One finds that $(\underline{C}_f)(k)$ is essentially the suspension of the set $\{j\in\mathbb{N}\mid k\ge j\}$ with the structure maps given by inclusion. Of course this means that $\underline{C}_f(k)$ has the same homotopy type as $\bigvee_{j\ge k} S_j^1$. It is easy to find a level map

$$\phi:\underline{S}^1\to\underline{C}_f$$

such that each $\phi(k)$ is a homotopy equivalence. In HoTop, ϕ is thus an isomorphism, i.e. $[\Sigma^{r-1}\underline{C}_f, \underline{X}]$ and $[\underline{S}^r, \underline{X}]$ are naturally isomorphic. (To convince oneself of this the reader is urged to draw some pictures!).

The map in the sequence from $\underline{C}_f$ to $\underline{S}^1$ identifies the two suspension points of $\underline{C}_f$ to the base point of $\underline{S}^1$. Thus it is easily seen that the induced morphism

$$\begin{array}{c} [\underline{S}^r, \underline{X}] \to [\Sigma^{r-1}\underline{C}_f, \underline{X}] \\ \hphantom{[\underline{S}^r, \underline{X}] \to} \cong \\ \hphantom{[\underline{S}^r, \underline{X}] \to} [\underline{S}^r, \underline{X}] \qquad (r \geq 2) \end{array}$$

is obtained as the difference of two mappings, the identity and the shift,

$$S_r : \underline{S}^r \to \underline{S}^r$$

which sends S_j^r homeomorphically onto S_{j+1}^r. For $r = 1$, it is $\mathrm{Id}_* S^{-1}$ and for $r = 0$ it is undefined.

(Again diagrams will help here.)

Thus we have a long exact sequence

$$\cdots \to \underline{\underline{\pi}}_n(\underline{X}) \xrightarrow{\alpha_n} \underline{\pi}_n(\underline{X}) \xrightarrow{\beta_n = \mathrm{Id} - S} \underline{\underline{\pi}}_n(\underline{X}) \xrightarrow{\gamma_n} \underline{\underline{\pi}}_{n-1}(\underline{X}) \to \cdots$$

and each map has a precise geometric description.

Remark. It is of interest to look at the geometric significance of the sequence in proper homotopy. Here it is the inclusion of Brown's $\underline{S}^0$ into a pair of lines $S^0 \times \mathbb{R}^+$ that replaces the folding map. The proof that a long cofibration sequence results is however hard to give directly.

We summarise this construction in the following theorem.

Theorem. *Given any pointed simplicial set $\underline{X}$, there is an exact couple*

$$\begin{tikzcd} \underline{\pi}_*(\underline{X}) \arrow[rr, "\beta_*"] & & \underline{\pi}_*(\underline{X}) \arrow[dl, "\gamma_*"] \\ & \underline{\underline{\pi}}_*(\underline{X}) \arrow[ul, "\alpha_*"] & \end{tikzcd}$$

linking the Steenrod and Brown–Grossman homotopy groups of $\underline{X}$. For $n \geq 2$, β_n is $\mathrm{Id} - S_n$ *where S_n is the shift map in dimension n, β_1 is* $\mathrm{Id}_* S_1^{-1}$. *α is induced by the folding map $\bigvee S^n \to S^n$ and γ_* has a degree -1.*

Corollary. *For any pointed simplicial $\underline{X}$, there is an exact couple*

$$\begin{tikzcd} \underline{\pi}^s_*(X) \arrow[rr, "\beta_*"] & & \underline{\pi}^s_*(X) \arrow[dl, "\gamma_*"] \\ & \underline{\underline{\pi}}^s_*(X) \arrow[ul, "\alpha_*"] & \end{tikzcd}$$

where $\underline{\pi}_*^s$ denotes the stable Brown–Grossman homotopy groups and $\underline{\underline{\pi}}_*^s$ the stable Steenrod homotopy groups, and for any spectrum E an exact couple

$$\begin{tikzcd}
{}_E\underline{h}_*(X) \arrow[r, "\beta_*"] & {}_E\underline{h}_*(X) \arrow[dl, "\gamma_*"] \\
{}_E\underline{\underline{h}}_*(X) \arrow[u, "\alpha_*"] &
\end{tikzcd}$$

where ${}_E\underline{h}_*$ denotes the Brown–Grossman E-homology and ${}_E\underline{\underline{h}}_*$ the Steenrod E-homology. In each case $\beta_* = \mathrm{Id} - S_*$, S_* a shift map, α_* is a folding map and $\deg \gamma_* = -1$.

4. Analysis

In the case that $\underline{X}$ is indexed by the ordered natural numbers (that is in the case of a prospace associated to a metric compactum, or to an end of a locally finite infinite complex) we can use the calculations of Grossman [9] to obtain an algebraic analysis of the various morphisms in the exact couples.

Firstly we recall that in such a case

$$\underline{\pi}_n(\underline{X}) = \varprojlim I(\pi_n(\underline{X}))$$

where I is the reduced product functor. With this identification the shift map in $\underline{\pi}_n(\underline{X})$ occurs as the map induced from the shift $S: I \to I$. (Recall that $I(G)$ is a quotient of the product and the shift on the product clearly induces a shift on $I(G)$.) For any given index k,

$$\mathrm{Id} - S(k): I(\pi_n(\underline{X}(k))) \to I(\pi_n(\underline{X}(k)))$$

has kernel the diagonal map $\Delta: \pi_n(X(k)) \to I(\pi_n(X(k)))$ and $\mathrm{Id} - S(k)$ is itself an epimorphism. Thus we have a short exact sequence of progroups

$$0 \to \pi_n(\underline{X}) \to I(\pi_n(\underline{X})) \xrightarrow{\mathrm{Id} - S} I(\pi_n(\underline{X})) \to 0.$$

Applying $\varprojlim$ and using the fact (Grossman [9]) that $\varprojlim^{(1)} \overline{I(\underline{G})} = 0$ for all towers of groups, $\underline{G}$, one has a four term exact sequence

$$\begin{array}{ccccccc}
0 \to \check{\pi}_n(\underline{X}) & \longrightarrow & \underline{\pi}_n(\underline{X}) & \xrightarrow[\beta_*]{\qquad} & \underline{\pi}_n(\underline{X}) & \longrightarrow & \varprojlim^{(1)} \pi_n(\underline{X}) \to 0 \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \\
\varprojlim \pi_n(\underline{X}) & & \varprojlim I(\pi_n(\underline{X})) & \xrightarrow{\lim(\mathrm{Id} - S)} & \varprojlim I(\pi_n(\underline{X})). & &
\end{array}$$

Thus we obtain from the exact couple a family of exact sequences

$$0 \to \check{\pi}_n(\underline{X}) \to \underline{\pi}_n(\underline{X}) \xrightarrow[\beta_n]{} \underline{\pi}_n(\underline{X}) \xrightarrow[\gamma]{} \underline{\underline{\pi}}_{n-1}(\underline{X}) \xrightarrow[\alpha_{n-1}]{} \check{\pi}_{n-1}(\underline{X}) \to 0. \qquad (1)$$

(This is the form of the exact couple found by Quigley [15].) Moreover we can split this into two shorter sequences since we know $\operatorname{Coker}_{\beta_v} = \varprojlim^{(1)} \pi_n(X)$.

$$0 \to \check{\pi}_n(\underline{X}) \to \underline{\pi}_n(\underline{X}) \xrightarrow{\beta_n} \underline{\pi}_n(\underline{X}) \to \varprojlim^{(1)} \underline{\pi}_n(\underline{X}) \to 0 \tag{2}$$

and

$$0 \to \varprojlim^{(1)} \pi_n(\underline{X}) \to \underline{\underline{\pi}}_{n-1}(\underline{X}) \to \check{\pi}_{n-1}(\underline{X}) \to 0. \tag{3}$$

The second of these sequences is, of course, the 'Milnor–exact sequence for Steenrod homotopy'. Stabilising and using E-homology will give stable homotopy and E-homology forms of this result. Several of these forms appear in the other papers of Quigley but his proofs are done elementwise, hence are longer.

It should be noted that the geometric picture given by the cofibration sequence tells us quite a lot about the geometric nature of these sequences.

If $\underline{X}$ is not indexed by the natural numbers, our methods are no longer valid. For instance we no longer have that

$$\varprojlim^{(1)} I(\pi_n(\underline{X})) = 0$$

but one can argue geometrically that $\ker \beta_n = \check{\pi}_n(\underline{X})$ so we still have the exact sequence (1).

We can split the sequence (1) in two parts as before to obtain forms of (2) and (3) with $\varprojlim^{(1)} \pi_n(X)$ replaced by $\operatorname{Coker} \beta_n$ and can give a geometric interpretation of $\operatorname{Coker} \beta_n$ and of the maps in the sequences. At present however calculation of this Cokernel seems out of the question. Sequence (2) is also part of the Bousfield–Kan spectral sequence so one may be able to find $\operatorname{Coker} \beta_n$ by this means.

Turning finally to the exact couple itself we can attempt to calculate derived couples etc. (cf. Hilton and Stammbach [10]) to see if this gives additional information. Firstly we notice that the composite $\gamma_* \alpha_*$ is induced by the equatorial map $S^{n-1} \to S^n$ in dimension n and hence is trivial. Since this is the d used in the calculation of the derived couple, we have that in the passage from

$$\begin{tikzcd} D \arrow[rr, "\beta"] & & D \arrow[dl, "\gamma"] \\ & E \arrow[ul, "\alpha"] & \end{tikzcd} \qquad \gamma\alpha = 0$$

to the derived couple,

$$\begin{tikzcd} \beta D \arrow[rr, "\beta'"] & & \beta D \arrow[dl, "\gamma'"] \\ & E^1 \arrow[ul, "\alpha'"] & \end{tikzcd}$$

$E^1 = H(E,d) \cong E$, (i.e. for such a derived couple with $\gamma\alpha = 0$, the E^∞ term is just E itself). We thus obtain an exact sequence [10; p. 280]

$$u_*(\underline{X}) \xrightarrow{\beta} u_*(\underline{X}) \xrightarrow{\gamma^\infty} \underline{\underline{\pi}}_*(\underline{X}) \xrightarrow{\alpha^\infty} J_*(\underline{X}) \xrightarrow{\beta^*} J_*(\underline{X})$$

where $u_*(\underline{X}) = \underline{\underline{\pi}}_*(\underline{X})/\bigcup \ker(\beta_*)^n$ and $J_*(\underline{X}) = \bigcap_n \beta_*^n(\underline{\pi}_*(X))$. Since β is 'Id-(Shift)', we can give a geometric interpretation of these groups, however again calculation would seem to be difficult.

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